Chapter 2:
Matrix Algebra

Sec. 2.2-2.3: Matrix Multiplication

Multiplying a Row by a Column

<u>Here's how to multiply a row matrix by a column matrix…</u>

<u>Ex 1</u>: Multiply

$$\begin{bmatrix} 2 & -1 & -4 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 5 \\ -2 \end{bmatrix}$$

Multiplying Matrices

If A is an $m \times n$ matrix and B is a $p \times q$ matrix, to find AB…

- The number of columns of A must equal the number of rows of B (that is, $n = p$)
- The size of the result will be $m \times q$
- To get the (i, j)-th entry of the answer, multiply the ith row of A with the jth row of B

Ex 2:

Compute the $(1, 3)$- and $(2, 4)$-entries of AB where

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix}.$$

Then compute AB.

Multiplying Matrices

Ex 3:

If $A = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix}$, compute A^2, AB, BA, and B^2 when they are defined.

Multiplying Matrices

Ex 4:

Let $A = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$. Compute A^2, AB, BA.

Multiplying Matrices

If A is an $m \times n$ matrix and B is a $p \times q$ matrix, to find AB…

- The number of columns of A must equal the number of rows of B (that is, $n = p$)
- The size of the result will be $m \times q$
- To get the (i, j)-th entry of the answer, multiply the ith row of A with the jth row of B
- **Formula: If $\boldsymbol{A = [a_{ij}]}$ and $\boldsymbol{B = [b_{ij}]}$, then $\boldsymbol{AB = \left[\ \sum_{k=1}^{n} a_{ik} b_{kj}\ \right]}$**

The Identity Matrix I

What does an IDENTITY really mean? (numbers/matrices)

The identity matrix is the square matrix with 1's on the main diagonal and 0's everywhere else. Sometimes we write I_n for the $n \times n$ identity matrix.

Ex 5: $I_3 =$

Verify $I_2 \begin{bmatrix} 1 & 2 & 7 \\ 3 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 7 \\ 3 & 0 & 9 \end{bmatrix}$

Verify $\begin{bmatrix} 1 & 2 & 7 \\ 3 & 0 & 9 \end{bmatrix} I_3 = \begin{bmatrix} 1 & 2 & 7 \\ 3 & 0 & 9 \end{bmatrix}$

Properties of Matrix Multiplication

Theorem 2.3.3

Assume that a is any scalar, and that A, B, and C are matrices of sizes such that the indicated matrix products are defined. Then:

1. $IA = A$ and $AI = A$ where I denotes an identity matrix.
2. $A(BC) = (AB)C$.
3. $A(B+C) = AB+AC$.
4. $(B+C)A = BA+CA$.
5. $a(AB) = (aA)B = A(aB)$.
6. $(AB)^T = B^T A^T$.

Prove some of these…

Properties of Matrix Multiplication

Ex 6: Simplify the expression $A(BC - CD) + A(C - B)D - AB(C - D)$.

Properties of Matrix Multiplication

Ex 7: Suppose that A, B, and C are $n \times n$ matrices and that both A and B commute with C; that is, $AC = CA$ and $BC = CB$. Show that AB commutes with C.

Properties of Matrix Multiplication

Ex 8: Show that $AB = BA$ if and only if $(A-B)(A+B) = A^2 - B^2$.

Special Properties of Matrices

1) In the matrix product AB, if you indicate B's columns, then…

Illustrate with $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 5 & 2 \\ 3 & 0 & 4 \end{bmatrix}$

Special Properties of Matrices

2) In the matrix product AB, if you indicate A's rows, then…

Illustrate with $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 5 & 2 \\ 3 & 0 & 4 \end{bmatrix}$

Special Properties of Matrices

3) If $\vec{x}$ is a column vector, the matrix product $A\vec{x}$ is a linear combination of the columns of A and the coefficients are the corresponding entries in $\vec{x}$.

Illustrate with $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 2 \\ 7 & 4 & 2 \\ 8 & 6 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$

Systems of Equations as a Matrix Equation

Every system of linear equations can be written as the matrix equation $A\vec{x} = \vec{b}$ where…

A is the matrix of coefficients of the variables in the system

$\vec{x}$ is the column matrix of variables

$\vec{b}$ is the column matrix of the constant terms in the system

Illustrate with

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 1 \\ 2x_1 - 4x_2 + x_3 \phantom{{}+3x_4} &= 5 \\ x_1 - 2x_2 + 2x_3 - 3x_4 &= 4 \end{aligned}$$

Why?

Systems of Equations as a Matrix Equation

Ex 9: Solve the matrix equation $A\vec{x} = \vec{b}$ by row reducing an augmented matrix.

$$A = \begin{bmatrix} 1 & -2 & 5 \\ -4 & 3 & 7 \\ 2 & -6 & 9 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 2 \\ 9 \\ -1 \end{bmatrix}$$

Results From Section 1.3: Homogeneous Systems of Linear Equations

Results: Suppose C_1 and C_2 are 2 solutions to a homogeneous system of linear equations written as column matrices, and suppose a_1 and a_2 are scalars. Then…

- $C_1 + C_2$ is also a solution to the system (sum of solutions is also a solution)
- $a_1 C_1$ is also a solution to the system (scalar multiple of a solution is another solution)
- $a_1 C_1 + a_2 C_2$ is also a solution to the system (linear combinations of solutions are solutions)

Proof: